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HEREDITARY NOETHERIAN CONDITIONS

DISSERTATION

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Preface

Some properties of rings, particularly some which characterize its ideal structure can be inherited by the ideals of the ring itself, regarded as subrings of the original one.

Prof. L. Budach proposed to me to study the noetherian rings such that all its ideals are also noetherian. We call these rings in this paper strong noetherian rings.

In particular it is easy to prove that an ideal I in a noetherian ring A is noetherian if and only if the ring $Z' + I$, where Z' is the image of Z in A by the canonical homomorphism, is noetherian. A natural generalization of this case is to regard A as B-algebra over a noetherian ring B and look for the necessary and sufficient conditions for $B' + I$ to be a noetherian ring B' denoting the image of B by the structural B-algebra homomorphism. The rings A such that $B' + I$ is noetherian for all ideals of A are called B-strong noetherians.

In the same frame of ideas arises also in a natural way the question, when a given noetherian A-module is also noetherian regarded as $(B + I)$-modul for every ideal I of a given set of ideals W? One of the chapters of this work considers such $(B + I)$-noeth categories. These are very close related to the filter of ideals of A, $F_I = \{J : A/I+J \in B\text{-finite}\}$, in such a way that $M \in (B + I)$-noeth if and only if $\operatorname{ann} M \in F_I$.

Many of the results which come about considering B instead of Z are easy generalizations due to the noetherian condition imposed on B and the categorical character of the method. Any way, we prefered to state them explicitly although no special considerations are given to the ring B, except when B is just Z. The last chapter is dedicated to a somewhat more complete study of the strong noetherian domains.

It is perhaps interesting to remark how in the noetherian properties of a ring play the prime ideals an important role, in this sense the well known result that every ideal in a ring has a finite base if and only if every prime ideal does, is in a certain way generalized by our result that a noetherian ring A is B-strong noetherian if and only if $B + P$ is noetherian for

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every $P \in \operatorname{Spec} A$, hence, a noetherian ring is strong noetherian if and only if all its prime ideals are noetherian.

I wish to thank Prof. Dr. L. Budach under whose guidance this paper came to live, for his constant stimulating interest and encouragement. I am also indebted to Prof. Dr. H. Kurke for his interest and suggestions.

Berlin, november 1972

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0. Conventions

We shall adopt the following conventions:

A ring will always mean a commutative ring, and except when the contrary is specifically stated, with a unit element.

If R is any ring we denote by:

M(R) :	the set of all ideals of R
R-mod :	the category of R-modules
R-noeth :	the category of noetherian R-modules

By a finite R-module we mean a finitely generated R-module; by a finite R-algebra we mean an algebra which is a finite R-module.

R-finite will denote the category of finite R-algebras.

Z' will represent the image of Z by the canonical homomorphism in a ring A.

If I is an ideal of a ring A, S a multiplicative closed set of A, I_S will denote the set: $\{a \in A : \exists s, \text{ such that } s\,a \in I\}$.

When P is a prime ideal of A we just write I_P for I_S, when $S = A \setminus P$.

Q (A) is the quotient field of a domain A.

Z and Q will represent as usual the ring of rational integers and the field of rationals, respectively.

ht (I) denotes the height of the ideal I.

A Grellian ring is an integral domain of dimension one.

1. Some general results

First we are going to state some general results which will be of some use later on.

1.1. Proposition: The subcategory R-noeth is dense in R-mod.

1.2. Proposition: Let $\mathcal{C}$ be a dense subcategory of R-mod, and M a finitely generated R-modul. Then $R \in \mathcal{C}$ implies $M \in \mathcal{C}$.

Proof: Since $\mathcal{C}$ is dense we can take M free, say $M = \bigoplus_{i=1}^{n} Rm_i$.

The exact sequence:

$$0 \longrightarrow \bigoplus_{i=1}^{n-1} Rm_i \longrightarrow M \longrightarrow M/\bigoplus_{i=1}^{n-1} Rm_i \longrightarrow 0$$

and the induction on the number of generators of M reduces the problem to proof that $Rm_n \in \mathcal{C}$. The canonical R-mod isomorphism $Rm_n \cong R/J_n$, $J_n = \operatorname{ann} m_n$ completes the proof.

1.3. Proposition: Let R be a noetherian ring, $\mathcal{C}$ a dense subcategory of R-mod and I a fixed ideal of R. Then the set of ideals:

$$F_I = \left\{ J \in H(R) : R/I + J \in \mathcal{C} \right\}$$

is an idempotent filter of R.

Proof: Let $J \in F_I$ and $J \subset J'$, then it follows $J' \in F_I$. This fact is clear from the exact sequence:

$$0 \longrightarrow I + J'/I + J \longrightarrow R/I + J \longrightarrow R/I + J' \longrightarrow 0$$

Assume now that $J_1, J_2 \in F_I$, then it holds also $J_1 J_2 \in F_I$

The inclusion $(I + J_1)(I + J_2) \subseteq I + J_1 J_2$, and what we have already proved above show that it is only necessary to prove the following assertion: For any two ideals $H_1, H_2 \in K(R)$, the condition $R/H_1, R/H_2 \in \mathcal{C}$ implies $R/H_1 H_2 \in \mathcal{C}$.

One has the exact sequence of R-modules

$$0 \longrightarrow H_1/H_1 H_2 \longrightarrow R/H_1 H_2 \longrightarrow R/H_1 \longrightarrow 0$$

Now, $R/H_1 \in \mathcal{C}$ by our assumption and $H_1/H_1 H_2$ is R/H_2-finite,

hence also $M_1/M_1M_2 \in \mathcal{C}$. Therefore $R/M_1M_2 \in \mathcal{C}$ by density of $\mathcal{C}$, qed.

1.4. Proposition: Let R be noetherian and let F be an idempotent filter of ideals of R, then it holds the equivalence:

$$J \in F \Longleftrightarrow V(J) \subset \Gamma \Longleftrightarrow \mathrm{Ass}(R/J) \subset \Gamma,$$

$$V(J) = \left\{ P \in \mathrm{Spec}\, R;\ P \supset J \right\}.$$

Proof: ($\Longrightarrow$) Trivial.

($\Longleftarrow$) Since R is noetherian there exist prime ideals $P_1, P_2, \ldots, P_s \in \mathrm{Ass}(R/J)$, and natural numbers $n_1, \ldots, n_s$ such that: $P_1^{n_1} \ldots P_s^{n_s} \subset J$; hence $P_1, \ldots, P_s \in \Gamma$ implies $J \in F$, qed.

2. Noetherian conditions in rings of the type B + I

Remark: In what follows, unless the contrary is specifically stated, both rings A and B, $A \supset B$, are assumed to be noetherians. I will denote an ideal of A.

2.1. Proposition: Let $A \supset B$ be noetherian rings and let I be a fixed ideal of A. Then it holds the equivalence:

$$A \in (B + I)\text{-noeth} \Longleftrightarrow A/I \in B\text{-finite}$$

Proof: ($\Longrightarrow$) $A \in (B + I)$-finite since it is noetherian as $(B + I)$-algebra, therefore $A/I \in B$-finite.

($\Longleftarrow$) It is enough to prove that $B + I$ is noetherian as ring (i. e. that $B + I \in (B + I)$-noeth), since the condition $A/I \in B$-finite implies that A is $(B + I)$-finite.
We need then only to prove that every ideal J' of $B + I$ is $(B + I)$-finite.
Let first J be any ideal of A, then J/IJ is A/I-finite, hence also B-finite. Thus J is $(B + I)$-finite.
Let now J' be any ideal of $B + I$. The product $J_1 = AJ'$ is an ideal of A and we have:

$$J_1/IJ_1 = AJ'/I(AJ') = AJ'/IJ' \cong J'/IJ'$$

Since $J_1/IJ_1 \in B$-finite and B is noetherian it follows that also $J'/IJ' \in B$-finite.
Our conclusion follows from the fact that IJ' is surely $(B + I)$-finite because is an ideal of A, qed.

2.2. Proposition: Let I be an ideal of a ring A, not assumed to be noetherian, and B a noetherian ring contained in A. Suppose that $I \in (B + I)$-noeth and $A/I \in B$-finite, then A is noetherian.

Proof: The structure of A/I as B-algebra coincides with its structure as $(B + I)$-algebra, therefore the assumption implies that $A/I \in (B + I)$-noeth. By density of $(B + I)$-noeth it follows $A \in (B + I)$-noeth, hence $A \in A$-noeth, qed.

2.2.1. Corollary: Let B be a noetherian ring contained in a ring A, not necessarily noetherian, and I an ideal of A. Then the following equivalence holds:

$I \in (B + I)\text{-noeth} \Longleftrightarrow B + I$ is noetherian

Proof: ($\Longleftarrow$) Trivial.

($\Longrightarrow$) Apply the above Proposition to $A = B + I$.

2.2.2. Remark: The ring $Z' + I$ is the smallest subring of A with unity which contains the ideal I.

The last corollary implies that the ideal I is noetherian (as ring, i. e. as $(Z' + I)$-modul) if and only if the ring $Z' + I$ is noetherian. In this way the problem to know when an ideal is noetherian is reduced to the study of such a problem for a ring with unit element.

2.3. Proposition: Let M be a finite A-module, then it holds the following equivalence:

$M \in (B + I)\text{-noeth} \Longleftrightarrow A/I + \operatorname{ann} M \in B\text{-finite}$

Proof: ($\Longleftarrow$) $A/I + \operatorname{ann} M \in B$-finite implies that $A \in (B + I + \operatorname{ann} M)$-noeth, therefore $M \in (B + I)$-noeth. ($\Longrightarrow$) Let $M = (Aa_1, \ldots, Aa_n)$; $J_i = \operatorname{ann} a_i$, $i = 1, \ldots, n$.

Then, $A/J_i \cong Aa_i$, and $M = \bigcap_{i=1}^{n} J_i$.

Aa_i is a submodule of M, thus $A/J_i \cong Aa_i \in (B + I)$-finite.

Hence $A/I + J_i \in B$-finite, for every $i = 1, 2, \ldots, n$.

The inclusion $(I + J_1)(I + J_2) \ldots (I + J_n) \subseteq I + \operatorname{ann} M$ and the Proposition 1.3. show that $A/I + \operatorname{ann} M \in B$-finite, qed.

2.3.1. Corollary: It holds the equivalence:

$I \in (B + I)\text{-noeth} \Longleftrightarrow B + I$ is noetherian $\Longleftrightarrow$

$\Longleftrightarrow A/I + \operatorname{ann} I \in B$-finite.

2.3.2. Remark: In the case when A is not taken to be necessarily a noetherian ring we have the following:

$A \in (B + I)$-noeth $\Longleftrightarrow A/I \in B$-finite, A noetherian

A noetherian $\Longleftarrow A/I \in B$-finite, $I \in (B + I)$-noeth

In general it doesn't follow from A noetherian and $I \in (B + I)$-noeth that $A/I \in B$-finite. Our corollary above asserts that only $A/I + \mathrm{ann} I \in B$-finite. (See example 2.4.2. below.)

We shall go on considering A to be noetherian. In this case it is probably convenient to state again explicitly our relations.

a) $A \in (B + I)$-noeth $\Longleftrightarrow A/I \in B$-finite

b) $I \in (B + I)$-noeth $\Longleftrightarrow B + I$ is noetherian $\Longleftrightarrow A/I + \mathrm{ann} I \in B$-finite

It is clear that from a) follows b). The following proposition gives a sufficient condition for the converse.

2.4. Proposition: Let $I_P \neq 0_P$ for every prime ideal P of A such that $ht(P) = 0$, then:

$B + I$ noetherian $\Longleftrightarrow A \in (B + I)$-noeth

Proof: Of course it is only necessary to prove the implication ($\Longrightarrow$). As a consequence of the Proposition 1.3, it is enough to show that all the elements of $\mathrm{ann} I$ are nilpotent, hence that $\mathrm{ann} I \subseteq \bigcap_{ht(P)=0} P$.

We have: $0 = (\bigcap_{ht(P)=0} Q_i) \cap M$, whence, $P \in \mathrm{Ass}(A/M)$ implies $ht(P) > 0$.

$$\mathrm{ann} I = [0 : I]_A = (\bigcap_{ht(P)=0} [0_P : I]_A) \cap [M : I]_A \subseteq$$
$$\subseteq \bigcap_{ht(P)=0} [0_P : I]_A$$

and also: $[0_P : I]_A = [0_P : I_P] \subseteq P$, since in general the equivalence $I_P \neq 0_P \Longleftrightarrow [0_P : I_P]_A \subseteq P$ is true for every $P \in \mathrm{Spec}\, A$.

Therefore, $\mathrm{ann} I \subseteq \bigcap_{ht(P)=0} P$, q.e.d.

2.4.1. Corollary: When I is a regular ideal or $ht(I) \geq 1$ it

holds the equivalences:
$B + I$ noetherian $\Longleftrightarrow A \in (B + I)$-noeth $\Longleftrightarrow$
$\Longleftrightarrow A/I \in B$-finite

Proof: In both cases $I_p \neq O_p$ for every $P \in \operatorname{Spec} A$ such that $ht(P) = 0$.

2.4.2. We are going now to give an example in which $Z' + I$ is noetherian but A is not $(Z' + I)$-noetherian, or in other words, such that $A/I + \operatorname{ann} I \in Z$-finite but $A/I \not\in Z$-finite.

Just take $A = Z \oplus Z[x]$, x and indeterminate, $I = 2 + (0)$. Then we have: $\operatorname{ann} I = 0 \oplus 2[x]$, $A/I + \operatorname{ann} I \cong 0$, $A/I \cong Z[x] \not\in Z$-finite.

In this case $I = P$ is prime, $ht(I) = 0$, and, $I = I_p = O_p$.

3. Categories of the type (B + I)-noeth

As a direct consequence of the Propositions 1.3. and 2.3. we can now obtain some results over finitely generated A-modules.

3.1. From Proposition 1.1. it follows that the category (B + I)-noeth of the A-modules which are noetherians (B + I)-modules is a dense subcategory of the category A-mod.

The filter $F_I = \{J : A/I + J \in B\text{-finite}\}$ is an idempotent filter since B-finite is a dense subcategory of A-mod, and the Proposition 2.3. is nothing else but the equivalence: $\operatorname{ann} M \in F_I \Longleftrightarrow M \in (B + I)\text{-noeth}$.

In general, given an idempotent filter F of ideals of the ring A, a dense subcategory $\mathcal{C}_F$ of A-mod is defined in the following way (see Gabriel [3]): $M \in \mathcal{C}_F \Longleftrightarrow \operatorname{ann} M \in F$
It is easy to prove that the category so defined is dense.
Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence.
Then, $\operatorname{ann} M' \supset \operatorname{ann} M$; $\operatorname{ann} M'' \supseteq [M' : M]_A \supset \operatorname{ann} M$; thus, $M \in \mathcal{C}_F \Longrightarrow M', M'' \in \mathcal{C}_F$. On the other hand from $\operatorname{ann} M' \cdot \operatorname{ann} M'' \subset \operatorname{ann} M$ it follows: $M', M'' \in \mathcal{C}_F \Longrightarrow M \in \mathcal{C}_F$. Our category (B + I)-noeth is the restriction of the category $\mathcal{C}_{F_I}$ to the subcategory of the finite A-modules.

3.2. As a simple application of the Proposition 1.4. to the filter F_I we can state:

$$M \in (B + I)\text{-noeth} \Longleftrightarrow \operatorname{ann} M \in F_I \Longleftrightarrow V(\operatorname{ann} M) \subset V_I \Longleftrightarrow \operatorname{Ass}(M) \subset V_I$$

3.3. Corollary: The following conditions are satisfied:

a) Let $J \supset I$ be ideals in A, then:
$(B + I)\text{-noeth} \subset (B + J)\text{-noeth}$

b) Let $J_1, \ldots, J_n$ be ideals in A, then:

$$\bigcap_{i=1}^{n} (B + J_i)\text{-noeth} = (B + \prod_{i=1}^{n} J_i)\text{-noeth} =$$

$$= (B + \bigcap_{i=1}^{n} J_i)\text{-noeth}$$

c) $(B + I)$-noeth $\Longleftrightarrow \bigcap_{P \in V(I)} (B + P)$-noeth $\Longleftrightarrow$

$$\Longleftrightarrow \bigcap_{P \in Ass(A/I)} (B + P)\text{-noeth}$$

Proof: One needs only to apply the properties of the filter F_{annM}.

3.4. Remark: One can consider the category $\bigcap_{I \in W} (B + I)$-noeth, $W \subseteq H(R)$ a set of ideals of A, and then it is clear that the above equivalence 3.3. c) holds also in the form.

$$\bigcap_{I \in W} (B + I)\text{-noeth} \Longleftrightarrow \bigcap_{P \in V(W)} (B + P)\text{-noeth} \Longleftrightarrow \bigcap_{P \in Ass(A/I),\, I \in W} (B + P)\text{-noeth}$$

with: $M \in \bigcap_{I \in W} (B + I)$-noeth $\Longleftrightarrow annM \in \bigcap_{I \in W} F_I = F_W$ (also an idempotent filter)

3.5. Corollary: Let I and J be ideals in A, then if $(B + I)$ and $(B + J)$ are noetherian, $(B + IJ)$ is also noetherian.

Proof: The ideal IJ is contained in $(B + I) \cap (B + J)$, hence $IJ \in (B + I)\text{-noeth} \cap (B + J)\text{-noeth}$. The result follows from 3.3. b).

3.6. Corollary: If $B + P$ is noetherian for every prime ideal $P \in V(I)$, then also $B + I$ is noetherian.

Proof: For every $P \in V(I)$ is I a $(B + P)$-modul and as such $(B + P)$-noetherian. From 3.3. c) $I \in (B + I)$-noeth, hence $B + I$ is noetherian.

3.7. Remark: The assertion of Corollary 3.5. implies that the set of ideals I of A such that $B + I$ is noetherian is a multiplicative set. However in general this set is not a filter, it can happens that although $B + I$ is noetherian there exists and ideal J which contains I and such that $B + J$ is not noetherian.

We consider the same example as in 2.4.2. $A = \mathbb{Z}[X_i \in \mathbb{Z}]$:

$I = 0 \oplus \mathbb{Z}$; $J = \mathbb{Z}g \oplus \mathbb{Z}$; where g is a non-monic irreducible polynomial of $\mathbb{Z}[x]$.

Then, of course, $\mathbb{Z}' + I$ (or say I) is noetherian, but since $A/J + \operatorname{ann} J \cong \mathbb{Z}[x]/(g) \not\subseteq \mathbb{Z}$ -finite, $\mathbb{Z}' + J$ is not noetherian.

This situation can however come about in certain cases as follows from the Proposition 2.4. We state the result in the following.

3.8. Corollary: Let $I_P \neq 0_P$ for every ideal $P \in \operatorname{Spec} A$ such that $\operatorname{ht}(P) = 0$, and $J \supset I$, then it holds:

$$B + I \text{ noetherian} \overset{(a)}{\Longleftrightarrow} B + P \text{ noetherian},$$

$$\forall P \in V(I) \overset{(b)}{\Longrightarrow} B + J \text{ noetherian}.$$

Proof: Enough to prove $(\overset{(a)}{\Longrightarrow})$. We have by Proposition 2.4. $B + I$ noetherian $\Longrightarrow A/I \in B$-finite $\Longrightarrow A/P \in B$-finite $\Longrightarrow$ $\Longrightarrow B + P$ noetherian, for every $P \in V(I)$. The implication $\overset{(b)}{\Longrightarrow}$ is trivial.

From the last result follows immediatly.

3.9. Corollary: If A is an integral domain, the set of ideals I such that $B + I$ is noetherian is an idempotent filter.

3.10. Remark: From the equivalence: $M \in (B + I)$-noeth $\Longleftrightarrow A/I + \operatorname{ann} \in B$-finite and the properties of the filters already considered one derives also the following:

$M \in (B + I)$-noeth $\Longleftrightarrow A/P_i + Q_j \in B$-finite, for every pair of prime ideals $P_i \in \operatorname{Ass}(M)$; $Q_j \in \operatorname{Ass}(A/I)$.

It is easy to realise that no one of the conditions $A/P_i \in B$-finite, $\forall P_i \in \operatorname{Ass}(M)$, or $A/Q_j \in B$-finite, $\forall Q_j \in \operatorname{Ass}(A/I)$ is necessary for M to be $(B + I)$-noeth.

3.11. Remark: Let M be as above a finite A-module, $A' = A/\operatorname{ann} M$ and $f : A \to A'$ the canonical homomorphism. Put $I' = f(I)$, $B' = f(B)$. Then, $M \in (B + I)$-noeth $\Longleftrightarrow A/I + \operatorname{ann} M \in B$-finite $\Longleftrightarrow$ $A'/I' \in B'$-finite $\Longleftrightarrow A' \in (B' + I')$-noeth.

The study of the categories of the type $(B + I)$-noeth is reduced then to the study of the rings of the type $A/\operatorname{ann} M$.

3.12. Remark: We have been considering rings $A \supset B$, both noetherians, and rings of the form $B + I$ where I is an ideal of A.

It is clear that in general it is only necessary to regard A as a B-algebra and to consider the rings $B' + I$, where B' is the image of B by the structural B-algebra homomorphism. For simplicity we take $B' = B \subset A$.

3.13. Remark: One can generalize some of the previous results by considering the concept of μ-noetherian modules introduced by L. Budach, for an arbitrary ordinal number μ.

It holds, for instance, the following equivalence:

$A \in \mu\text{-}(B + I)\text{-noeth} \iff A/I \in \mu\text{-}B\text{-noeth}$.

4. B-strong noetherian rings

In this chapter as before, A and B will be assume to be noetherians. B' denote the image of the structural B-algebra homomorphism.

4.1. Definition: A noetherian B-algebra A over a noetherian ring B is said to be B-strong noetherian, or symbolically $A \in B\text{-}(SN)$ if for every ideal I of A the ring $B' + I$ is noetherian.

For simplicity we shall take $B = B'$. When $B = Z$ we just simply say that A is strong noetherian or that $A \in (SN)$. This is the case we are going to consider more often.

4.1.1. Proposition: A ring A is B-strong noetherian if and only if for every prime ideal $P \in \operatorname{Spec} A$ the ring $B + P$ is noetherian, or equivalently, if and only if $A/P + \operatorname{ann} P \in B\text{-finite}$.

Proof: It is clear from Corollary 3.6.

4.1.2. Remark: From Corollaries 3.6. and 2.4.1. it follows that when A is an integral domain the following conditions are equivalent:

a) $A \in B\text{-}(SN)$

b) $A \in \bigcap_{I \in (H(A) \setminus 0)} (B + I)\text{-noeth}$

c) $A/I \in B\text{-finite}, \forall I \in (H(A) \setminus 0)$

d) $A/P \in B\text{-finite}, \forall P \in (\operatorname{Spec} A \setminus 0)$

For an integral domain A to be B-strong noetherian it is not necessary that $A \in B\text{-finite}$ as we shall see by examples.

As a direct consequence of Corollary 3.3. c) we have

4.1.3. Proposition: When A has zero divisors the following conditions are equivalent:

$$A \in \bigcap_{I \in (H(A) \setminus 0)} (B + I)\text{-noeth} \iff A \in \bigcap_{P \in \operatorname{Spec} A} (B + P)\text{-noeth}$$

$$\iff A \in B\text{-finite}$$

4.1.4. Remark: It is evident that: $A \in B\text{-finite} \Rightarrow A \in (B\text{-}(SN)$. However, the converse doesn't hold in general. This means that in this case the condition $A \in \bigcap_{I \in (H(A) \setminus 0)} (B + I)\text{-noeth}$ is actually more restrictive that the B-strong noetherian one

(see example 4.2.7.).

4.1.5. Proposition: Let A be B-strong noetherian, B not a field, then dim A $\leq$ dim B. When B is a field dim A $\leq$ 1.

Proof: We may assume without any loss of generality that dim A is finite and A is an integral domain, since dim A $=\infty$ implies dim B $=\infty$. In the finite dimensional case suppose the assertion holds for integral domains, then as there exist a minimal prime ideal P of A such that dim A = dim (A/P) we can write the inequalities: dim A = dim (A/P) $\leq$ dim (B/P$\cap$B) $\leq$ dim B.

So, let A be an integral domain which is B-strong noetherian. Define: Y = {set of minimal prime ideals which are the first terms ($\neq$ 0) of chains of prime ideals of A of maximal length, i. e. of length equal to dim A}

(1) $0 \subset P_0 \subset P_1 \subset \ldots \subset$

If for any element $P_0 \in Y$ it holds $P_0 \cap B \neq 0$, then since A/P_0 is $B/P_0 \cap B$-finite, one gets: dim A - 1 = dim (A/P_0) = dim $(B/P_0 \cap B) <$ dim B. Therefore dim A $\leq$ dim B.

We shall prove that such P_0 always exists. Let suppose it doesn't, then for every $P_0 \in Y$, $P_0 \cap B = 0$, hence dim (A/P_0) = dim B, (2).

Let (1) above be one of such chain of maximal length. We assert that $P_1 \cap B \neq 0$, because otherwise dim (A/P_1) = dim B contradicts (2).

Let $0 \neq a \in P_1 \cap B = Q_1$, since $Aa \subset P_1$, by the principal ideal theorem there exists a minimal prime ideal P_0' of A such that $Aa \subset P_0' \subset P_1$. Then $P_0' \in Y$ and, of course, $P_0' \cap B \ni a \neq 0$.

When B is a field and A is not, then it is clear that dim A = 1, since A/P is B-finite for every prime ideal $P \neq 0$ of A, hence P is maximal, qed.

4.1.6. Corollary: A ring A is strong noetherian if and only if dim A $\leq$ 1 and for every prime ideal P of A, the residue class ring A/P+annP $\in$ Z -finite.

4.1.7. Corollary: An integral domain A which is not a field is strong noetherian if and only if is a Grellian ring such that any one of the two following equivalent conditions hold:
a) All the residue class rings A/P, $P \in (\mathrm{Spec}\, A \setminus 0)$ are Galois fields.
b) All the residue class rings A/I, $I \in (H(A) \setminus 0)$ are finite.

4.1.8. Remark: When A has zero divisors the conditions a) and b) of the above corollary are in general not fulfilled. Nor even it holds that for every prime ideal P of A the ring A/P+annP is finite.
As an example take $A = Z[x]/(f^2)$, where f is a monic irreducible polynomial in Z[x]. The ideal $(\bar{f}) = (f)/(f^2)$ is prime in A, $\mathrm{ann}(\bar{f}) = (\bar{f})$ and $A/(\bar{f})+\mathrm{ann}(\bar{f}) \overset{\mathrm{def}}{\simeq} Z[x]/(f)$. It is evident that $A \in (SN)$ since A is even Z-finite but Z[x]/(f) is of course not a finite ring.

4.1.9. Remark: In the general case, i. e. when D is different from Z, dim A can be any natural number. Take, for instance, $A = C[x_1,\ldots,x_n]$, $B = Z[x_1,\ldots,x_n]$, where $C \in Z$-finite and the x_i are indeterminates. A is B-finite and for this reason B-(SN). Moreover dim A = dim B = n.

4.2. Before we start a more systematic study of the strong noetherian rings, particularly of the strong noetherian domains, we shall derive some results for B-(SN)-rings. The proves work exactly as in the case of (SN)-rings but it is convenient to state them explicitly. The results for (SN)-rings come out as corollaries.

4.2.1. Proposition: Let A be B-(SN) and I any ideal of A, then (B + I) is also B-(SN).

Proof: Let J be any ideal of (B + I), we have to prove that B + J is noetherian. The equivalence $J \in (B + J)$-noeth $\Longleftrightarrow$ B + J noetherian shows that it is enough to prove that every ascending chain:

(1) $J_1 \subseteq J_2 \subseteq \ldots \subseteq J$ of $(B + J)$-modules in J, becomes stationary.

We have also the following ascending chain:

(2) $IJ \cap J_1 \subseteq IJ \cap J_2 \subseteq \ldots \subseteq IJ$ of $(B + IJ)$-modules

This chain becomes stationary since IJ is an ideal of A. Therefore exists a natural number n such that:

$IJ \cap J_n = IJ \cap J_{n+1} = \ldots$; and for $i > n$ we can also write down the following ascending chain:

(3) $J_1/IJ \cap J_n \subseteq J_{i+1}/IJ \cap J_n \subseteq \ldots \subseteq J/IJ$

Since J is finite as $(B + I)$-module, J/IJ is B-finite, therefore the last chain (3) has a maximal element, hence the chain (1) too, qed.

4.2.2. Proposition: Let I be a fixed ideal in A; then the following condition is satisfied:
$(B + I) \in B\text{-}(SN)$, $A/I \in B\text{-}(SN) \Leftrightarrow A \in B\text{-}(SN)$

Proof: ($\Longleftarrow$) That $A/I \in B\text{-}(SN)$ is clear. From Proposition 4.2.1. follows also that $(B + I) \in B\text{-}(SN)$.
($\Longrightarrow$) We must prove that if J is any ideal of A, then $B + J$ is noetherian, or equivalently that every strictly ascending chain.

(1) $J_1 < J_2 < \ldots < J$ of $(B + J)$-modules in J is finite.

Consider the chain:
$I \cap J_1 \subseteq I \cap J_2 \subseteq \ldots \subseteq I \cap J$ of $(B + (I \cap J)$-modules, since $I \cap J$ is an ideal in $(B + I)$ and therefore $(B + (I \cap J)$ is noetherian, it holds $I \cap J_n = I \cap J_{n+1} = \ldots$ for $n \gg 0$.

Taking $i > n$ we have also the following chain:

(3) $J_i/I \cap J_n \subseteq J_{i+1}/I \cap J_n \subseteq \ldots \subseteq J/I \cap J$

But since A/I is B-(SN) the ideal $J/I \cap J$ is $(B/I \cap B + J/I \cap J)$-noetherian, hence the last chain becomes stationary. This fact implies that the chain (1) becomes stationary too.

qed.

4.2.3. Corollary: Let I be such that $I_p \neq D_p$ for every prime ideal P of A such that $ht(P) = 0$, then holds the following equivalence:

$A \in B\text{-}(SN) \Longleftrightarrow (B + I) \in B\text{-}(SN)$

Proof: Enough to prove ($\Longleftarrow$). Let J be any ideal of A, we must prove that B + J is noetherian. Since IJ is an ideal of B + I, B + IJ is noetherian. From Corollary 3.8. it follows that B + J is noetherian.

4.2.4. Remark: In general the condition $(B + I) \in B\text{-}(SN)$ is not sufficient for A to be B-SN. For instance if $A = \mathbb{Z} \oplus \mathbb{Z}[X]$, $I = \mathbb{Z} \oplus 0$, then $I \in (SN)$ but $A \notin (SN)$.

4.2.5. Remark: When we apply the above Proposition 4.2.2. and its Corollary 4.2.3. to the case of strong noetherian rings we get:

$I \in (SN)$, $A/I \in (SN) \Longleftrightarrow A \in (SN)$; $I \in (SN) \Longleftrightarrow (Z' + I) \in (SN)$

and when I satisfies the condition of Corollary 4.2.3., for instance, when I is regular or $ht(I) \geq 1$, then:

$$I \in (SN) \Longleftrightarrow A \in (SN)$$

In the above sense we could say that the category of strong noetherian rings is a sort of "dense" subcategory in the category of rings (with or without unit element).
If σ is any ring without unit element we can in a canonical way imbedd it in a ring say $\sigma[1]$ with a unit element. Then, of course, $\sigma \in (SN) \Longleftrightarrow \sigma[1] \in (SN)$.

4.2.6. Proposition: The category B-(SN) of B-algebras which are B-strong noetherians is closed under finite direct sums.

Proof: Enough to give the proof in the case $A = A_1 \oplus A_2$. The prime ideals of A are of the form $P_1 \oplus A_2$, $P_1 \in \mathrm{Spec}\ A_1$; or $A_1 \oplus P_2$, $P_2 \in \mathrm{Spec}\ A_2$.
Let, for instance, be $P = P_1 \oplus A_2$, then it follows $\mathrm{ann}P \supseteq (\mathrm{ann}P_1)_{A_1} \oplus 0$, and $A_1 \oplus A_2/(P_1 + (\mathrm{ann}P_1)_{A_1}) \oplus A_2 \cong A_1/P_1 + (\mathrm{ann}P_1)_{A_1} \in B\text{-finite}$, hence $A/P + \mathrm{ann}P \in B\text{-finite}$, i. e. B + P is noetherian. qed.

4.2.7. Remark: From the above proposition it is now easy to

give the example we talk about in Remark 4.1.4. of a strong noetherian ring with zero divisors which is not Z-finite.

Take $A = Z_p \oplus Z_p$, $P \in \operatorname{Spec} Z$; $Z_p = Z_{A \setminus P}$.

5. Strong noetherian domains

We remind that the strong noetherian domains are just those domains A such that for every prime ideal $P \neq 0$ the residue class ring A/P is a Galois field, or equivalently, such that for every ideal $I \neq 0$ the residue class ring A/I is finite.

5.1. **Proposition:** An integral domain A is strong noetherian if and only if for every multiplicative closed system S of A the quotient ring A_S is also strong noetherian.

Proof: Assume first that $A \in (SN)$ and let S be a m.c.s. in A. All the prime ideals of A_S are of the form PA_S for some $P \in \operatorname{Spec} A$ such that $P \cap S = \emptyset$ and it holds the isomorphism $A_S/PA_S \simeq S^{-1}(A/P)$. Hence, all the rings A_S/PA_S are Galois fields.
Consider now that all the quotient rings A_S are strong noetherians, and let P be any prime ideal of A. Define $S = A \setminus P$, then the above isomorphism shows that A/P is finite,

q.e.d.

5.1.1. **Remark:** It is convenient to remark that the above equivalence doesn't hold in general when A is not an integral domain. Consider for instance the example of 4.1.8., and take $S = A \setminus (\bar{f})$. Then $A_S/(\bar{f})A_S + \operatorname{ann}(\bar{f})A_S) \simeq A_S/(\bar{f})A_S \simeq$

$S^{-1}(\mathbb{Z}[X]/f) \supset \mathbb{Q}$, hence is not Z-finite. Therefore $A_S \notin (SN)$

5.2. **Proposition:** Let $A \in (SN)$ and K an arbitrary field. Then the ring $A \cap K$ is also strong noetherian.

Proof: Let $A \cap K = A'$, and let $0 \neq b \in A'$ be an arbitrary element of A'. It is enough to prove that A'/A'b is a finite ring.
First of all we shall show that $A'b = Ab \cap K$. The inclusion $A'b \subset Ab \cap K$ is clear. Let then, $c \in A$ be such that $cb = r \in K$, hence $c = rb^{-1} \in A \cap K = A'$.
Therefore, $A'b = Ab \cap K \cap A' = Ab \cap A'$ and we have the inclusion

$$A'/A'b \subseteq A/Ab$$

Since A/Ab is finite this completes the proof.

5.3. Proposition: Let $A \in (\overline{SN})$ and A' a ring such that $A \subset A' \subset Q(A)$, then $A' \in (SN)$ too.

Proof: It is well known that if A is a Grellian ring then also A' is Grellian whenever $A \subset A' \subset Q(A)$. Moreover it holds that for every ideal $J \neq 0$ of A' the residue class ring A'/J is A-finite. Therefore if $P \neq 0$ is any prime ideal of A' the residue field A'/P is $(A/P \cap A)$-finite, hence finite, q.e.d.

5.4. Proposition: Let $A \in (SN)$ and let L be a finite algebraic extension of Q(A). Then any ring A' such that $A \subset A' \subset L$ is strong noetherian too.

Proof: There exist always elements $b_1, \ldots, b_s \in L$ such that $A[b_1, \ldots, b_s] \subset A'$ and such that both rings have the same quotient field. By Proposition 5.3. above it is then enough to consider the case $A' = A[b_1, \ldots, b_s]$, and by induction on s, just the case $A' = A[b], b \in L$.

Now, $A' = A[b]$ is Grellian and for any prime ideal $P \neq (0) \neq A'$, the contracted ideal $P \cap A \neq 0$, thus A'/P is a Galois field.

q.e.d.

5.5. Proposition: Let $A \in (SN)$, L a finite algebraic extension of Q(A) and $A' \subset L$ integral over A. Then it holds the following equivalence:
$A \in (SN) \Leftrightarrow A' \in (SN)$

Proof: ($\Longrightarrow$) follows from Proposition 5.4.
($\Longleftarrow$) Let $A' \in (SN)$ and let Q be any prime ideal of A.
By the "going-up" theorem there exist a prime ideal P of A' such that $P \cap A = Q$. The inclusion $A/Q \subset A'/P$ shows that A/Q must be finite, q.e.d.

5.6. We shall now prove some general theorems about the strong noetherian domains.

5.6.1. Proposition: Let B be a Grellian ring, then the following conditions are equivalent:

a) $B \in (SN)$

b) $B = \bigcap_{i \in T} R_i$; where the R_i are local domains of dimension 1 such that $R_i \subset Q(B)$, $R_i \in (SN)$.

c) $B' = \bigcap_{i \in T'} R'_i$; B' is the integral closure of B; the R'_i are discrete valuation rings such that $R'_i \subset Q(B)$, $R_i \in (SN)$.

Proof: a)$\Longrightarrow$b). Is clear that $B = \bigcap_{P \in Spec B} B_P$, that $B_P \in (SN)$ for all P, and that all these rings are local domains of dimension one.

b)$\Longrightarrow$a) Let Q_i denote the maximal ideal of R_i for every $i \in T$. Since $R_i \in (SN)$ then $Q_i \cap B = P_i \neq 0$ and we have $B_{P_i} \subset (R_i)_{Q_i} = R_i$ for every $i \in T$. Therefore $B = \bigcap_{i \in T} B_{P_i}$.

The P_i are now all the prime ideals of B (see Cohen lemma [8]) thus it is enough to prove that all the B/P_i are finite. The assertion follows easily from the fact $B/P_i \subseteq R_i/Q_i$.

From Proposition 5.5. and the above already established equivalence a)$\Longleftrightarrow$b) it is now easy to see that we actually have:

b)$\Longleftrightarrow$a)$\Longleftrightarrow B' \in (SN) \Longleftrightarrow$c) , qed .

5.6.2. Proposition: The intersection of a finite number of strong noetherian domains is strong noetherian.

Proof: First of all from Proposition 5.2. it follows that we can assume that all the rings have the same quotient field, namely the quotient field of the intersection.

Let $B = \bigcap_{i=1}^{n} B_i$, $\quad B_i \in (SN)$, $i = 1,2,\ldots,n$

Now B is Krullian (we could for instance use the above quoted lemma of Cohen), hence we can apply the Proposition 5.6.1. to the B_i. It is then clear that also B satisfies the hypothesis of the proposition, qed.

5.6.3. Proposition: A semilocal domain A of dimension one is strong noetherian if and only if it is the intersection of finite many strong noetherian local domains.

Proof: That the condition is necessary is clear from the fact that if $P_1, \ldots, P_r$ are the prime ideals of A, then

$A = \bigcap_{i=1}^{r} A_{P_i}$ and all the A_{P_i} are strong noetherians.

To prove sufficiency let $A = \bigcap_{i=1}^{n} A_i$ be an intersection of strong noetherian local domains (A_i, Q_i). By Proposition 5.6.2, $A \in (SN)$ and moreover the $P_i = Q_i \cap A$, $i = 1, \ldots, n$ are all the prime ideals of A, qed.

5.6.4. Proposition: Let A be an integral domain and let x be transcendental over A. Then A[x] is strong noetherian if and only if A is a finite field.

Proof: Trivial.

5.7. Some examples: We are now in position to give some examples of strong noetherian domains.

1. $Z \in (SN)$, since $Z/(p)$ is a Galois field for every rational prime p.
2. Every ring in an algebraic number field i. e. every ring A such that $[Q(A) : Q] < \infty$
3. Every ring in an algebraic function field over a Galois field i. e. rings A such that $[Q(A) : F_p(x)] < \infty$. F_p denotes the finite field of p elements.
4. All discrete valuation rings (A, m) such that A/m is finite. For instance the completions $Z_{(p)}$ with respect to the p-adic valuations of Q, or the valuation rings of the form $F_p[[x]]$.
5. Rings in the local fields, i. e. rings contained in the finite algebraic extensions of Q_p (field of p-adic

numbers) or of $Q(F_p[[X]])$.

The first case corresponds to the case of characteristic null, the second one to the characteristic p case.

5.7.1. We consider now some more examples which could contribute to give us some better knowledge of these rings.

1. Char (A) = o

a) There exist strong noetherian domains A with $tr(Q(A) : Q) = n$ for every natural number n.

Proof: Take in the ring $Z_{(p)}$ of p-adic integers n different transcendental elements over Q, say $t_1, \ldots, t_n$. Then $A = Z_{(p)} \cap Q(t_1, \ldots, t_n)$ is also strong noetherian and $Q(A) = Q(t_1, \ldots, t_n)$.

b) There exist strong noetherian domains A algebraic over Q with quotient field of infinite degree over Q. Take $A = Z_{(p)} \cap \overline{Q}$; $\overline{Q}$ denotes the algebraic closure of Q.

2. Char (A) = p ≠ o

a) There exist strong noetherian domains A with $tr(Q(A) : F_p(x)) = n$ for every natural number n.

Proof: Similar as in 1 a). Take n tanscendental elements $t_1, \ldots, t_n$ of $F_p[[X]]$ over $F_p(x)$ and consider $A = F_p[[X]] \cap F_p(x)(t_1, \ldots, t_n)$.

There exist strong noetherian domains A algebraic over $F_p(x)$ with quotient field of infinite degree over $F_p(x)$. Take $A = F_p[[X]] \cap \overline{F_p(x)}$

5.8. The problem to know whether the completion of a strong noetherian local domain is strong noetherian is reduced by the following proposition to determine whether or not the integral closure A' of a strong noetherian local domain A is A-finite.

5.8.1. Proposition: The completion $\overline{A}$ of a strong noetherian local domain A is strong noetherian if and only if A is analitically unramified (i. e. $\overline{A}$ is reduced). A is analitically unramified if and only if its integral

closure A' is A-finite.

Proof: The last assertion is well known, it holds for local domain of dimension one (see [12], for instance).
We shall prove then the first assertion. First of all we shall show the equivalence:

$\bar{A}$ reduced $\longleftrightarrow$ $\mathrm{ann}P \not\subseteq P$, for every minimal prime ideal P of $\bar{A}$.

Let $0 = O_{P_1} \cap \ldots \cap O_{P_r}$ be the irredundant decomposition of 0.

Our assertion follows from the equivalence:
$\mathrm{ann}P_i \not\subseteq P_i \longleftrightarrow [O_{P_i} : P_i] \not\subseteq P_i \longleftrightarrow O_{P_i} = P_i$, for all $i = 1, \ldots, r$.

The next step is to prove that $\bar{A}$ is strong noetherian if and only if $\mathrm{ann}P_i \not\subseteq P_i$ for every one of its minimal prime ideals.

Since $\dim \bar{A} = 1$ it is clear that $\mathrm{ann}P_i \not\subseteq P_i$ implies that $(\mathrm{ann}P_i + P_i)$ is $\mathfrak{M}$-primary, hence $\bar{A}/\mathrm{ann}P_i + P_i$ is finite, thus $\bar{A}$ strong noetherian.

Assume now that $\bar{A}$ is strong noetherian. Let $\bar{A}' = \prod_{i=1}^{n} \bar{A}'_{m_i}$ be the representation of the completion of the integral closure A' of A as direct product of the local domains $\bar{A}'_{m_i}$. The ideals $m_i, i = 1, \ldots, n$ are the maximal ideals of the semilocal domain A', the rings $\bar{A}'_{m_i}$ are discret valuation rings which are strong noetherian.

Consider the diagram:

$$\begin{array}{ccc} \bar{A} & \xrightarrow{f} & \bar{A}' = \prod_{i=1}^{n} \bar{A}'_{m_i} \\ & \searrow^{g_i} & \downarrow \mathrm{pr} \\ & & \bar{A}'_{m_i} \end{array}$$

(f the canonical map, pr the projection)

$\ker g_i = P_i$ is a minimal prime ideal of $\bar{A}$.

Since $\bar{A}$ is strong noetherian, $\mathrm{ann}P_i \not\subseteq P_i$ implies that $\bar{A}/P_i$ is a finitely generated group, but this is impossible unless $\bar{A}/P_i$ is a field. This contradiction proves our assertion.

5.8.2. We are going now to give two examples, one for characteristic null, the other for characteristic $p \neq 0$, of strong noetherian local domains A whose integral closure are not finite A-modules, therefore such that their completions

are not strong noetherian.

a) Example of Cohen - Zariski (see [9])

Let F be an arbitrary field, let L = F(x,y) be a field of rational functions of two independent variables, and let $u(x) = b_1 x + b_2 x^2 + \ldots, b_i \in F$ be a power series which is trascendental over F(x).

By substituting y = u(x) in elements of L we obtain a discrete valuation v of L. This valuation is actually the restriction of the x-adicvaluation of $F[[x]]$ to F(x,p(x)). Then, of course, v(x) = 1 and if G is the valuation ring its maximal ideal is Gx and every element of G is congruent to an element of F modulo Gx, hence $F \cong G/Gx$.
Let $R' = F(x,y^n) \cap G$, where n is a positive integer, so that R' is the valuation ring of the induced valuation in $L' = F(x,y^n)$, has R'x as its maximal ideal and F as its residue field, i. e. $F \cong R'/R'x$.

The ring $R'[y] = R$ has L as quotient field and is integrally dependent on R', because $y^n \in R'$. Hence, if $P \neq 0$ is a prime ideal in R, then $0 \neq P \cap R'$ is a prime ideal in R', hence contains x and also y^n since $y^n \in R'$, and therefore y. Thus $P \supseteq Rx + Ry$, and moreover P = Rx + Ry since $R/Rx + Ry \cong R'/R'x \cong F$, and P=Rx+Ry is maximal. Hence P is the only prime ideal of R different from the zero ideal, thus R is a local domain of dimension one.

We now assume that Char F = p >0 and that $n = p^e$ is a power of p. Then G is the integral closure of R and G is not a finite R-module.

Namely if $a \in G$, then $a^n \in L'$ and hence $a^n \in L \cap G = R'$; G is thus integrally dependent on R' and hence on R.

If G were a finite R-module there would exist an element $b \neq 0$ in R such that $bG \subseteq R$. Since every ideal ($\neq 0$) in R is primary for P, some power of x is a multiple of b. Hence for some k, $x^k G \subseteq R$. But this is impossible. For

$c = (y - b_1 x - \ldots - b_k x^k)/x^{k+1}$ is certainly in G, but $x^k c = x^{-1} y + d$, $d \in R'$, and since elements of R are uniquely of the form $\sum_{i=0}^{n-1} c_i y^i$, $c_i \in H_1$, $x^k c \notin R$. qed.

b) Example of Akizuki (see [2])

In his example Akizuki constructs a local domain of dimension one (I, P') which satisfies the following condition:
The congruence $x^2 \equiv o \pmod{P'^M}$ with $x \not\equiv o \pmod{P'^e}$ for a fixed natural number e and any natural number M admits always a solution in I.

It is not difficult to show that this condition holds if and only if I is analitically ramified, i. e. if and only if the integral closure of I is not a finite I-modul.

Let call Z' the image of Z in Q_3 by the canonical homomorphism. One can always choose a p-adic integer:

(1) $a = a_1 p + a_2 p^3 + a_3 p^7 + \ldots + a_r p^{2^r - 1} + \ldots$

transcendental over Z', with the natural numbers a_i such that $1 \leq a_i \leq p - 1$.

Let:

(2) $u_1 = a/p$, $u_2 = u_1 - a_1/p^2, \ldots, u_{r+1} = u_r - a_r/p^{2^r}$

Each of the numbers $(u_r - a_r)$, $r = 1, 2, \ldots$, is a p-adic integer of the form:

$$(3) \quad \begin{cases} u_1 - a_1 = p^2 (a_2 + a_3 p^4 + a_4 p^{12} + \ldots) \\ u_2 - a_2 = p^4 (a_3 + a_4 p^8 + a_5 p^{24} + \ldots) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ u_r - a_r = p^{2^r} (a_{r+1} + a_{r+2} p^{2^{r+1}} + \ldots) \end{cases}$$

We extend Z' by adjoining to it the following sequence

of p-adic integers:

(4) $a,\ i_1 = (u_1 - a_1)^2,\ i_2 = (u_2 - a_2)^2, \ldots,$

$i_r = (u_r - a_r)^2 \ldots$

Let:

(5) $R = Z'[a,\ i_1,\ i_2, \ldots, i_r, \ldots]$

Then for every $r > 1$ we have

(6) $i_{r-1} = (u_{r-1} - a_{r-1})^2 \equiv o \pmod{pR}$

Since by (2) and (3):

$(u_{r-1} - a_{r-1})^2 = (p^{2^{r-1}} u_r)^2 = p^{2^r} [a_r + (u_r - a_r)]^2 =$

$= p^{2^r} [a_r^2 + 2a_r (u_r - a_r) + (u_r - a_r)^2]$

and,

$p^{2^r} (u_r - a_r) = p^{2^{r+1}} [a_{r+1} + a_{r+2} p^{2^{r+1}} + \ldots]$ (by (3))

$= p [a - (a_1 p + \ldots + a_r p^{2^r - 1})]$

hence we get:

(7) $i_{r-1} = p^{2^r} (a_r^2 + i_r) + 2a_r p [a - (a_1 p + a_2 p^3 + \ldots$

$\ldots + a_r p^{2^r - 1}] \equiv o \pmod{pR}$

But one has:

(8) $a \not\equiv o \pmod{p R}$, i. e. $u_1 \notin R$

because otherwise u_1 would be a polynomial in finite many elements of (A) with coefficients in Z', say:

$$u_1 = f(a, i_1, \ldots, i_j)$$

$$u_1 = F(a, i_j) = F(pu_1, (u_j - a_j)^2),$$

since $i_j = (u_j - a_j)^2$ by (7) can be expressed as a linear function of i_1, for every $j > 1$.

Since u_1 is transcendental with a over Z' this equation

must be an identity when we consider u_1 as an indeterminate. From (3) we get:

$$u_1 = a_1 + a_2p^2 + a_3p^6 + \ldots + a_1p^{2^1-2} + p^{2^1-2}(u_1 - a_1)$$

and the above identity can be written.

$$(9)\quad p^{2^1-2}u_1 = G(p^{2^1-1}u_1, (u_1 - a_1)^2),$$

where $G(t_1,t_2)$ represents a polynomial in t_1 and t_2 over Z'.

But this is impossible, as one can easily check by making say the change of variables $u_1 - a_1 = y$ and comparing the coefficients.

Moreover;

$$(10)\quad a^2 = \left[pa_1 + p(u_1 - a_1)\right]^2 = -a_1^2p^2 + 2a_1pa + p^2i_1$$

$$\equiv 2a_1pa \pmod{p^2R},$$

the ideal $P = (p,a)$ is maximal, since from (6)

$$R/P \cong Z'/pZ'$$

and $P \neq pR$ by (8)

From (10) one gets also:

$$(12)\quad P^2 = (p^2,pa,a^2) = p(p,a) = pP.$$

One considers now the quotient ring $I = R_P$, which is local.

Let;

$$b_r = a - (a_1p + a_2p^3 + \ldots + a_rp^{2^r-1}) \qquad r = 0,1,2\ldots$$

then from (3) $\quad b_r = p^{2^r-1}(u_r - a_r)$

and from (7): $b_r^2 = p^{2^{r+1}-2}(u_r - a_r)^2$

$$= p^{2^{r+1}-2}\left[p^{2^{r+1}}(a_{r+1}^2 + i_{r+1}) + 2a_{r+1}p(b_r - a_{r+1}p^{2^{r+1}-1})\right]$$

$$= p^{2^{r+1}-2}\left[p^{2^{r+1}}(-a_{r+1}^2 + i_{r+1}) + 2a_{r+1}pb_r\right]$$

hence,

$$(13)\quad b_r^2 \equiv 2a_{r+1}p^{2^{r+1}-1}b_r \pmod{p^{2^{r+1}}R}$$

Furthermore:

(14) $b_r \not\equiv o \ (P^2 I)$

because:

$b_r \in P^2 I \Rightarrow b_r \in P^2 R = P^2 I \cap R \Rightarrow b_r \in pR \Rightarrow a \in p R$ which is a contradiction to (8).

One shows now using the above expressions that every principal ideal in I contains some power of PI, hence that every ideal is PI-primary, therefore that I is a primary domain.

Since PI has a base I is noetherian.

Now one shows that the congruence $x^2 \equiv o \ (\text{mod } (PI)^M)$ with $x \not\equiv o \ (\text{mod } (PI)^2)$ admits always a solution in I for every M. It is enough to take r big enough since from (14) $b_r \not\equiv o \ (P^2 I)$ and from (13) $b_r \equiv o \ (\text{mod } (IP)^{2r+1})$, which are true for all r, qed.

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